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COHOMOLOGY OF THE STEENROD ALGEBRA

W.-H. Lin*

Department of Mathematics, National Tsing Hua University, Hsinchu, Taiwan 300, Republic of China

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1. Introduction

Let A denote the mod 2 Steenrod algebra. Let $h_i \in \mathrm{Ext}_A^{1,2^i}(\mathbb{Z}_2,\mathbb{Z}_2)$ be the classes corresponding to the generators $\mathrm{Sq}^{2^i} \in A$ as described by Adams in [2]. D.M. Davis shows in [5] that h_i are acted on faithfully by portions of $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_2,\mathbb{Z}_2)$ which increase with i. More precisely, he shows that if $\alpha \neq 0$ in $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2,\mathbb{Z}_2)$ with $0<t-s<2^j$, then $\alpha h_i \neq 0$ for $i \geq 2j+1$. In this paper we prove a similar result. We prove h_i^2 are acted on faithfully by portions of $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_2,\mathbb{Z}_2)$ which increase with i. To state precisely the result we fix some notation. Let A_l be the sub-Hopf-algebra of A generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \ldots, \mathrm{Sq}^{2^l}$. The set $\{n \mid \exists a \neq 0 \text{ in } A_l \text{ such that } |a|=n\}$ is bounded where $|a|$ means $\deg(a)$. Let d_l be the largest integer in this set. We will show later that $d_l=(l-1)2^{l+2}+l+5$.

Theorem 1.1. *Let α be a non-zero class in $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2,\mathbb{Z}_2)$ with $t-s>0$. Let i be the smallest integer such that $2^i-2 \geq t-s$. Then $\alpha h_m^2 \neq 0$ for all m such that $2^{m-1}>sd_{i+1}-t$.*

Corollary 1.2. *$h_{i_1}^2 h_{i_2}^2 \cdots h_{i_n}^2 \neq 0$ in $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_2,\mathbb{Z}_2)$ for any finite increasing sequence $\{i_1, i_2, \ldots i_n\}$ of positive integers such that the successive numerical conditions in Theorem 1.1 are satisfied.*

It is a conjecture [18] that the classes h_i^2 survive the Adams spectral sequence for the stable homotopy groups of spheres [1]. This conjecture is known to be true for $0 \leq i \leq 5$. If the conjecture is true, then the classes in (1.2) probably also survive the Adams spectral sequence. These problems, however, remain to be done.

Theorem 1.1 stems from a conjecture of Mahowald in [7] (Conjecture V.2.4); in particular it shows that a large part of Mahowald's conjecture is true. We refer to Mahowald's memoir [7] for the significance of his conjecture in homotopy.

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The following relations hold in $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_2,\mathbb{Z}_2)$:

$$(1)\ h_{i+1}h_i=0,\qquad (2)\ h_{i+2}^2h_i=0,\qquad (3)\ h_{i+1}^3=h_{i+2}h_i^2,$$
$$(4)\ h_0^{2^{i+1}}h_{i+1}=0,\qquad (5)\ h_0^{2^i}h_{i+2}^2=0,\qquad (6)\ h_i^2h_{i+3}^2=0.$$

The first four of these are due to J.F. Adams [2,3] and the rest are due to J.P. May [10]. It has been a conjecture that these are the only relations among the h_i's. Davis [5] has given an evidence for the conjecture by showing that these relations are closed under the squaring operations

$$\mathrm{Sq}^i:\mathrm{Ext}_A^{k,j}(\mathbb{Z}_2,\mathbb{Z}_2)\to\mathrm{Ext}_A^{k+i,2j}(\mathbb{Z}_2,\mathbb{Z}_2)$$

of Liulevicius [6]. From relations (3) we see any non-zero monomial α in the h_i's can be uniquely expressed as $\alpha=h_0^{\varepsilon_0}h_{i_1}^{\varepsilon_1}h_{i_2}^{\varepsilon_2}\cdots h_{i_n}^{\varepsilon_n}$ where $0<i_1<i_2<\cdots<i_n$, $\varepsilon_0\geq 0$ and $\varepsilon_j=1$ or 2 for $j\leq 1$. Theorem 1.1 shows that monomials of this form are non-zero provided $\varepsilon_0=0$ and the integers i_j are far apart from one another, which is a part of the conjecture.

Our proof of Theorem 1.1 is based on a spectral sequence of Adams [2]. In Section 2 we describe this spectral sequence and study some of its properties in the case which is not discussed in [2]. In Section 3 we make some calculations in the Steenrod algebra which arise when using the spectral sequence of Adams. In Section 4 we complete the proof of Theorem 1.1.

2. A spectral sequence of Adams

Let Γ a be connected, locally finite Hopf algebra over $\mathbb{Z}_2$, Λ a sub-Hopf-algebra of Γ, and $\bar\Gamma$ and $\bar\Lambda$ the augmentation ideals of Γ and Λ respectively. Let $\Omega=\Gamma/\Gamma\cdot\bar\Lambda$ and $\bar\Omega=\bar\Gamma/\bar\Gamma\cdot\bar\Lambda$. Λ acts on Ω and $\bar\Omega$ from the left via the inclusion $\Lambda\to\Gamma$. Let $F(\Gamma^*)$ be the cobar construction of Γ. We filter it by setting

$$[\alpha_1\,|\,\alpha_2\,|\cdots|\,\alpha_s]\in F(\Gamma^*)^{(p)}=F^{(p)}$$

if α_i annihilates $\bar\Lambda$ for at least p values of i. So $F(\Gamma^*)=F^{(0)}\supset F^{(1)}\supset\cdots$.

Theorem 2.1 (Adams). *This filtration of* $F(\Gamma^*)$ *defines a spectral sequence* $\{E_r^{p,q}\}$ *which converges to* $\mathrm{Ext}_\Gamma^{*,*}(\mathbb{Z}_2,\mathbb{Z}_2)$ *and one has*

$$E_1^{p,q}=H^{p+q}(F^{(p+1)})\cong\mathrm{Ext}_\Lambda^q((\bar\Omega)^p,\mathbb{Z}_2).$$

Here the superscripts "$p+q$" and "q" refer to homological degrees and

$$(\bar\Omega)^p=\begin{cases}\mathbb{Z}_2 & \text{if } p=0,\\ \underbrace{\bar\Omega\otimes\cdots\otimes\bar\Omega}_{p} & \text{if } p>0.\end{cases}$$

We recall a part of Adams' proof of Theorem 2.1. We begin by considering the vector-space dual of the spectral sequence $\{E_r^{p,q}\}$. Let $\bar{B}(\Gamma)$ be the bar construction of Γ. We filter it by setting

$$[a_1 | a_2 | \cdots | a_s] \in \bar{B}(\Gamma)^{(p)} = \bar{B}^{(p)}$$

if $a_i \in \bar{\Lambda}$ for at least $s-p$ values of i. Then $F^{(p)} = (\bar{B}(\Gamma)/\bar{B}^{(p-1)})^*$. Thus the resulting spectral sequence $\{E^r_{p,q}\}$ of this filtration on $\bar{B}(\Gamma)$ is the $\mathbb{Z}_2$-dual of $\{E_r^{p,q}\}$. It suffices to show

$$E^1_{p,q} = H_{p+q}(\bar{B}^{(p)}/\bar{B}^{(p-1)}) \cong \mathrm{Tor}_q(\mathbb{Z}_2, (\bar{\Omega})^p).$$

Adams proves this by considering certain subquotient complexes of the bar resolution $\Gamma \otimes \bar{B}(\Gamma)$. Specifically he considers for each $p \geq 0$ the quotient

$$C^{(p)} = \Lambda \otimes \bar{B}^{(p)} + \Lambda \otimes \bar{B}^{(p-1)}/\Gamma \otimes \bar{B}^{(p-1)}.$$

It is easy to see that $C^{(p)} = \Lambda \otimes (\bar{B}^{(p)}/\bar{B}^{(p-1)})$; so $C_s^{(p)} = 0$ if $s<p$ where the suffix s refers to homological degree.

Lemma 2.2.

$$H_s(C^{(p)}) \cong \begin{cases} (\bar{\Omega})^p & (s=p), \\ 0 & (s \neq p). \end{cases}$$

The isomorphism for $s=p$ is obtained by projecting Λ to $\mathbb{Z}_2$ and $(\bar{\Gamma})^p$ to $(\bar{\Omega})^p$.

Lemma 2.2 is Lemma 2.3.1 in [2] to which we refer for details of the proof.

Lemma 2.2 shows that the free Λ-complex $C^{(p)}$ is a free resolution of $(\bar{\Omega})^p$ over Λ where the Λ-action on $(\bar{\Omega})^p$ is determined by $C^{(p)}$ and 2.2. Thus

$$E^1_{p,q} = H_{p+q}(\bar{B}^{(p)}/\bar{B}^{(p-1)}) \cong H_{p+q}(\mathbb{Z}_2 \otimes_\Lambda C^{(p)}) = \mathrm{Tor}_q^\Lambda(\mathbb{Z}_2, (\bar{\Omega})^p).$$

This proves Theorem 2.1.

The action of Λ on $\bar{\Omega}$ is the usual one. For $p \geq 2$ the action of Λ on $(\bar{\Omega})^p$, however, is not the diagonal action. For our purpose it suffices to consider this action for $p=2$ which is described as follows. By the Milnor–Moore Theorem [12] Γ is free as a left or right module over Λ. Let $\{\gamma_i\}_{i\geq 0}$ be a right Λ-base for Γ with $\gamma_0 = 1$. Let $\bar{\gamma}_i$ be the image of γ_i in Ω. Then $\{\bar{\gamma}_i\}_{i\geq 1}$ is a $\mathbb{Z}_2$-base for $\bar{\Omega}$. Given $a \in \Lambda$ and $\bar{\gamma}_p \otimes \bar{\gamma}_p \in (\bar{\Omega})^2$, let $a\gamma_p = \sum_{\lambda=1}^n \gamma_{j(\lambda)} a_{j(\lambda)}$ with $a_{j(\lambda)} \in \Lambda$. Then

$$a(\bar{\gamma}_p \otimes \bar{\gamma}_q) = \sum_{\lambda=1}^n \bar{\gamma}_{j(\lambda)} \otimes a_{j(\lambda)} \bar{\gamma}_q. \tag{1}$$

This formula is derived from Adams' proof of Lemma 2.2 in [2]. A conceptually simple way to describe this action is the following. Γ, and hence Λ, acts from the left on $\Gamma \otimes_\Lambda \bar{\Omega} \cong \Omega \otimes \bar{\Omega}$ in a natural way. Then $(\bar{\Omega})^2$ is a Λ-submodule of $\Omega \otimes \bar{\Omega}$. This Λ-action on $(\bar{\Omega})^2$ can be shown to be isomorphic to the diagonal action (see (2.1) in [4]).

Remark 2.3. In [2] Adams discusses his spectral sequence only for the case that Λ is central in Γ, i.e., $ab=ba$ for all $a\in\Lambda$ and $b\in\Gamma$ (to serve other purposes there). In this case $\bar{\Omega}$ (hence $(\bar{\Omega})^p$ for $p\geq 2$) gets trivial operations from Λ. It suffices to assume Λ is normal in Γ, i.e., $\Gamma\cdot\bar{\Lambda}=\bar{\Lambda}\cdot\Gamma$ in order to have $\bar{\Omega}$ get trivial operations from Λ. Here we do not impose either condition on Λ as in our applications of Theorem 2.1 we shall take Γ to be the Steenrod algebra A and $\Lambda=A_l$ for some l where the A_l's are as described in Section 1 and these subalgebras are not normal in A.

To apply Theorem 2.1 in proving Theorem 1.1 we need to study the complexes $F^{(1)}/F^{(2)}$ and $F^{(2)}/F^{(3)}$. Consider the cobar constructions $F(\Lambda^*)\otimes(\bar{\Omega}^*)^p$, $p=1,2$. Our result (Proposition 2.4) is that there are a natural embedding

$$f_1: F(\Lambda^*)\otimes\bar{\Omega}^*\to F^{(1)}/F^{(2)}$$

and a projection

$$g_2: F^{(2)}/F^{(3)}\to F(\Lambda^*)\otimes(\bar{\Omega}^*)^2$$

such that both are chain equivalences. The map g_2 is not natural; it depends on the choice of a right Λ-base for Γ. It is possible to show that $F^{(p)}/F^{(p+1)}$ is chain equivalent to $F(\Lambda^*)\otimes(\bar{\Omega}^*)^p$ for any p. For our purpose we will only consider f_1 and g_2. Explicit formulae describing f_1 and g_2 will be relevant. It suffices to describe their $\mathbb{Z}_2$-duals

$$\bar{f}_1: \bar{B}^{(1)}/\bar{B}^{(0)}\to\bar{B}(\Lambda)\otimes\bar{\Omega} \quad\text{and}\quad \bar{g}_2:\bar{B}(\Lambda)\otimes(\bar{\Omega})^2\to\bar{B}^{(2)}/\bar{B}^{(1)}.$$

We begin with $\bar{f}_1$. For $a\in\Gamma$ let $\bar{a}$ be its image in $\bar{\Omega}$. Given $[a_1|\cdots|a_s]\in\bar{B}^{(1)}/\bar{B}^{(0)}$, there is a unique a_i such that $a_i\notin\bar{\Lambda}$. Then define $\bar{f}_1$ by

$$\bar{f}_1([a_1|\cdots|a_{s-1}|a_s])=\begin{cases}[a_1|\cdots|a_{s-1}]\otimes\bar{a}_s & (i=s),\\ 0 & (i<s).\end{cases} \tag{2}$$

$\bar{g}_2$ is a little complicated to describe. We choose a right Λ-base $\{\gamma_i\}_{i\geq 0}$ for Γ with $\gamma_0=1$. Then $\{\bar{\gamma}_i\}_{i\geq 1}$ is a $\mathbb{Z}_2$-base for $\bar{\Omega}$. We first define a $\mathbb{Z}_2$-map $\phi: M\to M$ where $M\subset\bar{B}^{(1)}/\bar{B}^{(0)}$ is generated by all $[a_1|\cdots|a_s]$ such that the unique $a_j\notin\bar{\Lambda}$ lies in $\{\gamma_i\}_{i\geq 1}$. Given $[a_1|\cdots|a_s]\in M$. Let a_j be the element such that $a_j=\gamma_k$ for some $k\geq 1$. We define $\phi([a_1|\cdots|a_s])$ by induction on j. If $j=1$, then set

$$\phi([\gamma_k|a_2|\cdots|a_s])=[\gamma_k|a_2|\cdots|a_s].$$

Suppose $j>1$ and suppose $\phi([a_1'|\cdots|a_s'])$ is defined for all $[a_1'|\cdots|a_s']$ such that the integer v for which $a_v'=\gamma_l$ is less than j. Let $a_{j-1}\gamma_k=\sum_{\lambda=1}^{n}\gamma_{t(\lambda)}a_{t(\lambda)}$ with $\gamma_{t(\lambda)}\in\{\gamma_i\}_{i\geq 0}$ and $a_{t(\lambda)}\in\Lambda$. By inductive hypothesis $\phi([a_1|\cdots|a_{j-2}|\gamma_{t(\lambda)}|a_{t(\lambda)}|a_{j+1}|\cdots|a_s])$ is defined for all λ. Then define

$$\phi([a_1|\cdots|a_{j-1}|\gamma_k|a_{j+1}|\cdots|a_s])=[a_1|\cdots|a_{j-1}|\gamma_k|a_{j+1}|\cdots|a_s]$$
$$+\sum_{\lambda=1}^{n}\phi([a_1|\cdots|a_{j-2}|\gamma_{t(\lambda)}|a_{t(\lambda)}|a_{j+1}|\cdots|a_s]).$$

Here $[c_1|\cdots|c_s]=0$ if $c_i=1$ for some i. This convention is also adopted in what follows.

We proceed to define $\bar{g}_2$. Given $[a_1|\cdots|a_s]\otimes(\bar{\gamma}_p\otimes\bar{\gamma}_q)\in\bar{B}(\Lambda)\otimes(\bar{\Omega})^2$, we define $\bar{g}_2([a_1|\cdots|a_s]\otimes(\bar{\gamma}_p\otimes\bar{\gamma}_q))$ by induction on s. If $s=0$, then set

$$\bar{g}_2(\bar{\gamma}_p\otimes\bar{\gamma}_q)=[\gamma_p|\gamma_q].$$

Suppose $s>0$ and suppose $\bar{g}_2([a_2|\cdots|a_s]\otimes(\bar{\gamma}_p\otimes\bar{\gamma}_q))$ is defined. Let

$$\sum_{\mu=1}^{l}[\gamma_{k(\mu)}|b_{2(\mu)}|\cdots|b_{s+1(\mu)}|\gamma_q]$$

be the sum of all those elements $[b'_1|\cdots|b'_{s+1}|b'_{s+2}]$ appearing in the sum $\bar{g}_2([a_2|\cdots|a_s]\otimes(\bar{\gamma}_p\otimes\bar{\gamma}_q))$ such that $b'_1\in\{\gamma_i\}_{i\geq1}$ and $b'_{s+2}=\gamma_q$ (if there is such a sum). Let $a_1\gamma_{k(\mu)}=\sum_\lambda\gamma_{t(\lambda,\mu)}a_{t(\lambda,\mu)}$ with $\gamma_{t(\lambda,\mu)}\in\{\gamma_i\}_{i\geq0}$ and $a_{t(\lambda,\mu)}\in\Lambda$. Then define

$$\bar{g}_2([a_1|a_2|\cdots|a_s]\otimes(\bar{\gamma}_p\otimes\bar{\gamma}_q))=[a_1|\bar{g}_2([a_2|\cdots|a_s]\otimes(\bar{\gamma}_p\otimes\bar{\gamma}_q))]$$
$$+\sum_{\lambda,\mu}[\gamma_{t(\lambda,\mu)}|\phi([a_{t(\lambda,\mu)}|b_{2(\mu)}|\cdots|b_{s+1(\mu)}|\gamma_q])]. \qquad (3)$$

Here if $\Theta=\sum[c_1|c_2|\cdots|c_{s+1}]$, then $[c|\Theta]$ denotes the sum $\sum[c|c_1|\cdots|c_{s+1}]$.

To give a more clear picture about the inductive formula (3) we explicitly write it out for the cases $s=1$ and $s=2$. For $s=1$, given $a\otimes(\bar{\gamma}_p\otimes\bar{\gamma}_q)\in\bar{B}(\Lambda)_1\otimes(\bar{\Omega})^2$, let $a\gamma_p=\sum_\lambda\gamma_{\lambda p}a_\lambda$ and let $a_\lambda\gamma_q=\sum_j\gamma_{jq\lambda}b_j$ with $\gamma_{\lambda p},\gamma_{jq\lambda}\in\{\gamma_i\}_{i\geq0}$ and $a_\lambda,b_j\in\Lambda$. Then

$$g_2([a]\otimes(\bar{\gamma}_p\otimes\bar{\gamma}_q))=[a|\gamma_p|\gamma_q]+\sum_\lambda[\gamma_{\lambda p}|a_\lambda|\gamma_q]+\sum_{\lambda,j}[\gamma_{\lambda p}|\gamma_{jq\lambda}|b_j].$$

For $s=2$, given $[a_1|a_2]\otimes(\bar{\gamma}_p\otimes\bar{\gamma}_q)\in\bar{B}(\Lambda)_2\otimes(\bar{\Omega})^2$, let

$$a_2\gamma_p=\sum_\lambda\gamma_{\lambda p}a_\lambda,\qquad a_\lambda\gamma_q=\sum_j\gamma_{jq\lambda}b_j,$$
$$a_1\gamma_{\lambda p}=\sum_\nu\gamma_{\nu\lambda p}c_\nu,\qquad c_\nu\gamma_{jq\lambda}=\sum_\mu\gamma_{\mu\nu jq\lambda}d_\mu$$

with $\gamma_{\lambda p},\gamma_{jq\lambda},\gamma_{\nu\lambda p},\gamma_{\mu\nu jq\lambda}$ in $\{\gamma_i\}_{i\geq0}$ and $a_\lambda,b_j,c_\nu,d_\mu$ in Λ. Then

$$\bar{g}_2([a_1|a_2]\otimes(\bar{\gamma}_p\otimes\bar{\gamma}_q))=[a_1|a_2|\gamma_p|\gamma_q]+\sum_\lambda[a_1|\gamma_{\lambda p}|a_\lambda|\gamma_q]$$
$$+\sum_{\lambda,j}[a_1|\gamma_{\lambda p}|\gamma_{jq\lambda}|b_j]+\sum_{\lambda,\nu}[\gamma_{\nu\lambda p}|c_\nu|a_\lambda|\gamma_q]$$
$$+\sum_{j,\lambda,\nu}[\gamma_{\nu\lambda p}|c_\nu|\gamma_{jq\lambda}|b_j]+\sum_{j,\lambda,\mu,\nu}[\gamma_{\nu\lambda p}|\gamma_{\mu\nu jq\lambda}|d_\mu|b_j].$$

Proposition 2.4. *The maps $f_1:F(\Lambda^*)\otimes\bar{\Omega}^*\to F^{(1)}/F^{(2)}$ and $g_2:F^{(2)}/F^{(3)}\to F(\Lambda^*)\otimes(\bar{\Omega}^*)^2$ with their $\mathbb{Z}_2$-duals $\bar{f}_1$ and $\bar{g}_2$ defined by* (2) *and* (3) *are chain equivalences.*

Proof. It suffices to show that $\bar{f}_1: \bar{B}^{(1)}/\bar{B}^{(0)} \to \bar{B}(A) \otimes \bar{\Omega}$ and $g_2: \bar{B}(A) \otimes (\bar{\Omega})^2 \to \bar{B}^{(2)}/\bar{B}^{(1)}$ are chain equivalences. Consider Adams free A-resolutions

$$C^{(p)} = A \otimes \bar{B}^p + \Gamma \otimes \bar{B}^{(p-1)}/\Gamma \otimes \bar{B}^{(p-1)} \cong A \otimes (\bar{B}^{(p)}/\bar{B}^{(p-1)})$$

and the bar resolutions $A \otimes \bar{B}(A) \otimes (\bar{\Omega})^p$ of $(\bar{\Omega})^p$. It is not difficult (although tedious) to verify that

$$1_A \otimes \bar{f}_1 : C^{(1)} \to A \otimes \bar{B}(A) \otimes \bar{\Omega} \quad \text{and} \quad 1_A \otimes \bar{g}_2 : A \otimes \bar{B}(A) \otimes (\bar{\Omega})^2 \to C^{(2)}$$

are chain maps over A and induce isomorphisms in homology. Thus both $1_A \otimes \bar{f}_1$ and $1_A \otimes \bar{g}_2$ are A-chain equivalences. So

$$\bar{f}_1 = 1_{\mathbb{Z}_2} \otimes_A 1_A \otimes \bar{f}_1 \quad \text{and} \quad \bar{g}_2 = 1_{\mathbb{Z}_2} \otimes_A 1_A \otimes \bar{g}_2$$

are chain equivalences. $\square$

We conclude this section by summarizing some properties of the map g_2 which follow immediately from formula (3).

We assume $\bar{A}$ is finite dimensional over $\mathbb{Z}_2$. Let d be the largest integer for which there are non-zero elements $a \in \bar{A}$ such that $|a| = d$. Let $\{v_\theta\}$ be a $\mathbb{Z}_2$-base for $\bar{\Gamma}$ such that $\{y_i\}_{i \le 1} \subset \{\theta_v\}$ and let $\{\theta_v^*\}$ be its dual base for $\bar{\Gamma}^*$. Note that $\bar{\Omega}^* \subset \bar{\Gamma}^*$. In the lemma below elements α_i of a non-zero cochain $[\alpha_1 | \cdots | \alpha_s]$ in $F(\Gamma^*)$ (or $F^{(p)}/F^{(p+1)}$) or non-zero elements in $\bar{\Omega}^*$ will be basis elements in $\{\theta_v^*\}$. Let $i^*: \bar{\Gamma}^* \to \bar{A}^*$ be the $\mathbb{Z}_2$-dual of the inclusion $i: \bar{A} \to \bar{\Gamma}$. We write

$$[\alpha_1 | \cdots | \alpha_s] \otimes (x \otimes y) \in g_2([\beta_1 | \cdots | \beta_{s+2}])$$

if $[\alpha_1 | \cdots | \alpha_s] \otimes (x \otimes y)$ appears in the sum $g_2([\beta_1 | \cdots | \beta_{s+2}])$.

Lemma 2.5. (i) *Suppose* $[\alpha_1 | \cdots | \alpha_s | \alpha_{s+1} | \alpha_{s+2}]$ *is a non-zero element in* $F^{(2)}/F^{(1)}$ *such that* $\alpha_{s+2} \in \bar{\Omega}^*$ *and* α_{s+1} *annihilates* $\bar{A}$. *Then*

$$g_2([\alpha_1 | \cdots | \alpha_s | \alpha_{s+1} | \alpha_{s+2}]) = \begin{cases} 0 & (\alpha_{s+1} \notin \bar{\Omega}^*), \\ [i^*(\alpha_1) | \cdots | i^*(\alpha_s)] \otimes (\alpha_{s+1} \otimes \alpha_{s+2}) & (\alpha_{s+1} \in \bar{\Omega}^*). \end{cases}$$

(ii) *Given* $[\alpha_1 | \cdots | \alpha_s] \in F(A^*)^{s,t}$ *and* $x, y, z \in \bar{\Omega}^*$, *let* $[\beta_1 | \cdots | \beta_{s+1}]$ *be an element in* $F^{(1)}/F^{(2)}$ *such that*

$$[\beta_1 | \cdots | \beta_{s+1} | z] \in (F^{(2)}/F^{(3)})^{s+2, t+|x|+|z|}.$$

If $z \ne y$, *then*

$$[\alpha_1 | \cdots | \alpha_s] \otimes (x \otimes y) \notin g_2([\beta_1 | \cdots | \beta_{s+1} | z]).$$

If $z = y$, $|x| > sd - t$ *and* $\sum_{j=1}^{s+1} |\beta_j| \le sd$, *then*

$$[\alpha_1 | \cdots | \alpha_s] \otimes (x \otimes y) \notin g_2([\beta_1 | \cdots | \beta_{s+1} | z]).$$

3. Some calculations in the Steenrod algebra which arise when using the spectral sequence of Adams

Let A_l be the sub-Hopf-algebra of the Steenrod algebra A generated by $\mathrm{Sq}^1, \mathrm{Sq}^2, \ldots, \mathrm{Sq}^{2^l}$ and let $\Omega = A/A \cdot \bar{A}_l$. In this section we determine the structure of Ω^* (Proposition 3.1) using Milnor's description of A and prove that $\bar{A}_{l-2}$ acts trivially on certain A_l-module generators of $(\bar{\Omega})^2$ (Proposition 3.4).

We begin by recalling from Milnor [11] that

$$A^* = \mathbb{Z}_2[\xi_1, \xi_2, \ldots]$$

and

$$A_l^* = \mathbb{Z}_2[\xi_1, \xi_2, \ldots]/(\xi_1^{2^{l+1}}, \xi_2^{2^l}, \ldots, \xi_{l+1}^2, \xi_{l+2}, \ldots)$$

with coproduct given by

$$\Delta(\xi_k) = \sum_{j=0}^{k} \xi_{k-j}^{2^j} \otimes \xi_j \qquad (\xi_0 = 1)$$

where $\deg(\xi_i) = 2^i - 1$. Let $\chi : A^* \to A^*$ be the canonical anti-automorphism of A^* [12] and let $\zeta_i = \chi(\xi_i)$. From the definition of χ we have

$$\zeta_1^n = \xi_1^n, \qquad \zeta_k = \xi_k + \sum_{j=1}^{k-1} \zeta_{k-j}^{2^j} \xi_j \quad (k \geq 2) \tag{4}$$

and

$$\Delta(\zeta_k) = \sum_{j=0}^{k} \zeta_j \otimes \zeta_{k-j}^{2^j}. \tag{5}$$

Then $A^* = \mathbb{Z}_2[\zeta_1, \zeta_2, \ldots]$ and

$$A_l^* = \mathbb{Z}_2[\zeta_1, \zeta_2, \ldots]/(\zeta_1^{2^{l+1}}, \zeta_2^{2^l}, \ldots, \zeta_{l+1}^2, \zeta_{l+2}, \ldots). \tag{6}$$

Let $\Omega = A/A \cdot \bar{A}_l$. Then $\Omega^* \subset A^*$.

Proposition 3.1. $\Omega^* = \mathbb{Z}_2[\zeta_1^{2^{l+1}}, \zeta_2^{2^l}, \ldots, \zeta_{l+1}^2, \zeta_{l+2}, \ldots]$.

This generalizes a result of F. Peterson in [13] where he proves Proposition 3.1 for $l = 1$. We shall follow Peterson's method to prove 3.1 and we begin by recalling a result of his in [13].

A acts on A^* from the left and from the right by transposing. More precisely, given $a \in A$ and $m^* \in A^*$, define am^* and m^*a by $\langle am^*, b\rangle = \langle m^*, ba\rangle$ and $\langle m^*a, b\rangle = \langle m^*, ab\rangle$. The operations of A lower the degrees.

Lemma 3.2 (Peterson). *Under the above A-action A^* is a left and a right algebra over A, that is, Cartan's formula holds and*

$$\mathrm{Sq}(\xi_k) = \xi_k + \xi_{k-1}^2, \qquad (\xi_k)\mathrm{Sq} = \xi_k + \xi_{k-1}$$

where $\mathrm{Sq}=\sum_{i=0}^{\infty}\mathrm{Sq}^i$.

It follows from Cartan's formula that

$$\mathrm{Sq}^{2^i}x^{2^j}=\begin{cases}(\mathrm{Sq}^{2^{i-j}}x)^{2^j} & (i\geq j),\\ 0 & (i<j)\end{cases}$$

for all $x\in A^*$.

Lemma 3.3. (i) $\mathrm{Sq}^{2^\lambda}\zeta_{k+2}=0$ *for* $\lambda\leq k$.
(ii) $\mathrm{Sq}^{2^\lambda}\zeta_k^{2^{l+2-k}}=0$ *for* $0\leq\lambda\leq l$ *and* $1\leq k\leq l+1$

Proof. We first deduce (ii) from (i). We may assume $\lambda\geq l+2-k$. Then

$$\mathrm{Sq}^{2^\lambda}\zeta_k^{2^{l+2-k}}=(\mathrm{Sq}^{2^{\lambda+k-l-2}}\zeta_k)^{2^{l+2-k}}=0$$

by (i) since $\lambda+k-l-2\leq k-2$.

We prove (i) by induction on k. If $k=0$, then $\lambda=0$ and $\zeta_{k+2}=\zeta_2=\xi_2+\xi_1^3$ (by (4)). We have

$$\mathrm{Sq}^1\zeta_2=\mathrm{Sq}^1\xi_2+\mathrm{Sq}^1\xi_1^3=\xi_1^2+\xi_1^2=0.$$

Thus the result is true for $k=0$. Suppose $k>0$ and suppose the result is true for $k'<k$. By (4)

$$\zeta_{k+2}=\xi_{k+2}+\sum_{j=1}^{k+1}\zeta_{k+2-j}^{2^j}\xi_j.$$

If $\lambda=0$, then

$$\begin{aligned}\mathrm{Sq}^1\zeta_{k+2}&=\mathrm{Sq}^1\xi_{k+2}+\sum_{j=1}^{k+1}(\mathrm{Sq}^1\zeta_{k+2-j}^{2^j})\xi_j+\sum_{j=1}^{k+1}\zeta_{k+2-j}^{2^j}\,\mathrm{Sq}^1\xi_j\\ &=\xi_{k+1}^2+\sum_{j=1}^{k+1}\zeta_{k+2-j}^{2^j}\xi_{j-1}^2\\ &=\xi_{k+1}^2+\sum_{i=1}^{k}\zeta_{k+1-i}^{2^{i+1}}\xi_i^2+\zeta_{k+1}^2\\ &=\left(\xi_{k+1}+\zeta_{k+1}+\sum_{i=1}^{k}\zeta_{k+1-i}^{2^i}\xi_i\right)^2=0\qquad\text{(by (4))}.\end{aligned}$$

If $\lambda\geq 1$, then

$$\begin{aligned}\mathrm{Sq}^{2^\lambda}\zeta_{k+2}&=\mathrm{Sq}^{2^\lambda}\xi_{k+2}+\sum_{j=1}^{k+1}\mathrm{Sq}^{2^\lambda}(\zeta_{k+2-j}^{2^j}\xi_j)\\ &=0+\sum_{j=1}^{k+1}(\mathrm{Sq}^{2^\lambda-1}\zeta_{k+2-j}^{2^j})\xi_{j-1}^2+\sum_{j=1}^{k+1}(\mathrm{Sq}^{2^\lambda}\zeta_{k+2-j}^{2^j})\xi_j\\ &=\sum_{j=1}^{k+1}(\mathrm{Sq}^{2^{\lambda-1}-1}\mathrm{Sq}^{2^{\lambda-1}}\zeta_{k+2-j}^{2^j})\xi_{j-1}^2+\sum_{j=1}^{\lambda}(\mathrm{Sq}^{2^\lambda}\zeta_{k+2-j}^{2^j})\xi_j\end{aligned}$$

$$= \sum_{j=1}^{\lambda-1} (\mathrm{Sq}^{2^{\lambda-1}-1}\mathrm{Sq}^{2^{\lambda-1}}\zeta_{k+2-j}^{2^j})\xi_{j-1}^2 + \sum_{j=1}^{\lambda} (\mathrm{Sq}^{2^\lambda}\zeta_{k+2-j}^{2^j})\xi_j$$
$$= \sum_{j=1}^{\lambda-1} \mathrm{Sq}^{2^{\lambda-1}-1}(\mathrm{Sq}^{2^{\lambda-1-j}}\zeta_{k+2-j})^{2^j}\xi_{j-1}^2 + \sum_{j=1}^{\lambda} (\mathrm{Sq}^{2^{\lambda-j}}\zeta_{k+2-j})^{2^j}\xi_j.$$

By inductive hypothesis $\mathrm{Sq}^{2^{\lambda-1-j}}\zeta_{k+2-j} = \mathrm{Sq}^{2^{\lambda-j}}\zeta_{k+2-j} = 0$ since $\lambda-1-j<\lambda-j\le k-j<k$. So $\mathrm{Sq}^{2^\lambda}\zeta_{k+2}=0$. This proves Lemma 3.3.

Proof of Proposition 3.1. For $a\in A$ let $R(a):A\to A$ and $L(a):A^*\to A^*$ be the maps defined by $R(a)b=ba$ and $L(a)b^*=ab^*$. Consider the exact sequence

$$\underbrace{A\oplus\cdots\oplus A}_{l+1} \xrightarrow{R(\mathrm{Sq}^1)\oplus\cdots\oplus R(\mathrm{Sq}^{2^l})} A \xrightarrow{\pi} (A/A\cdot\bar{A}_l) = \Omega \longrightarrow 0.$$

Dualizing this we get an exact sequence

$$\underbrace{A^*\oplus\cdots\oplus A^*}_{l+1} \xleftarrow{L(\mathrm{Sq}^1)\oplus\cdots\oplus L(\mathrm{Sq}^{2^l})} A^* \xleftarrow{\pi^*} \Omega^* \longleftarrow 0.$$

By Lemma 3.3 and Cartan's formula we see

$$\mathbb{Z}_2[\zeta_1^{2^{l+1}}, \zeta_2^{2^l}, \ldots, \zeta_{l+1}^2, \zeta_{l+2}, \ldots] \subset \ker \pi^* = \Omega^*.$$

But it is well known [9] that

$$\{\overline{\mathrm{Sq}(r_1, r_2, \ldots)} \mid r_i = \text{a multiple of } 2^{l+2-i} \text{ for } 0\le i\le l+2,\ r_{l+j}\ge 0 \text{ for } j\ge 2\}$$

is a $\mathbb{Z}_2$-base for Ω where $\mathrm{Sq}(r_1,r_2,\ldots)$ is the Milnor basis element in A dual to $\xi_1^{r_1}\xi_2^{r_2}\cdots$. Since $\deg(\xi_i)=\deg(\zeta_i)$, it follows that the vector spaces Ω^* and $\mathbb{Z}_2[\zeta_1^{2^{l+1}}, \zeta_2^{2^l}, \ldots, \zeta_{l+1}^2, \zeta_{l+2}, \ldots]$ have the same finite dimension in each degree and so they are equal. This proves Proposition 3.1.

We next proceed to show that $\bar{A}_{l-2}$ acts trivially on certain A_l-module generators of $(\bar{\Omega})^2$. We recall again that the Milnor basis for A is $\{\mathrm{Sq}(r_1,r_2,\ldots)\}$ which is dual to the monomial basis for the polynomial algebra $A^*=\mathbb{Z}_2[\xi_1,\xi_2,\ldots]$. Let $\chi:A\to A$ be the canonical anti-automorphism of A. Then $\{\chi\,\mathrm{Sq}(r_1,r_2,\ldots)\}$ is the basis for A, dual to the monomial basis for the polynomial algebra $A^*=\mathbb{Z}_2[\zeta_1,\zeta_2,\ldots]$. By Proposition 3.1 the set

$$\bar{B} = \{\overline{\chi\,\mathrm{Sq}(r_1,r_2,\ldots)} \mid r_i = k_i 2^{l-2-i} \text{ for } 0\le i\le l+2,\ r_{l+j}\ge 0 \text{ for } j\ge 2\} \quad (7)$$

is a $\mathbb{Z}_2$-base for Ω. We write $\mathrm{Sq}(r_2,r_2,\ldots,r_k)$ for $\mathrm{Sq}(r_1,r_2,\ldots)$ if $r_{k+j}=0$ for $j\ge 1$ and simply write $\chi(r_1,r_2,\ldots,r_k)$ for $\chi\,\mathrm{Sq}(r_1,r_2,\ldots,r_k)$.

Proposition 3.4. (i) $\overline{\chi(i\,2^{l+1})}\otimes\overline{\chi(j\,2^{l+1})}$ *are A_l-module generators of* $(\bar{\Omega})^2$ *where* $i>0$, $j>0$.

(ii) $\bar{A}_{l-2}$ *acts trivially on these generators* ($l\ge 2$).

Proof. By Lemma 3.3(ii) and Cartan's formula, $\mathrm{Sq}^{2^\lambda}\zeta_1^{i2^{l+1}}=0$ for $0\le\lambda\le l$. Since $\mathrm{Sq}^1,\mathrm{Sq}^2,\ldots,\mathrm{Sq}^{2^l}$ generate A_l it follows that $a\zeta_1^{i2^{l+1}}=0$ for $a\in\bar{A}_l$; so $\overline{\chi(i\,2^{l+1})}$ are A_l-module generators of $\bar{\Omega}$. Then formula (1) in Section 2 shows that $\overline{\chi(i\,2^{l+1})\otimes\chi(j\,2^{l+1})}$ are A_l-module generators of $(\bar{\Omega})^2$. This proves (i).

To prove (ii) we first show that $\bar{A}_{l-1}$ acts trivially on $\overline{\chi(j\,2^{l+1})}$, that is

$$a\overline{\chi(j\,2^{l+1})}=0 \quad \text{for } a\in\bar{A}_{l-1}. \tag{$*$}$$

It suffices to show that for any monomial

$$m=\zeta_{r_1}^{k_12^{l+2-r_1}}\zeta_{r_2}^{k_22^{l+2-r_2}}\cdots\zeta_{r_n}^{k_n2^{l+2-r_n}}$$

in $\Omega^*=\mathbb{Z}_2[\zeta_1^{2^{l+1}},\zeta_2^{2^l},\ldots,\zeta_{l+1}^2,\zeta_{l+2},\ldots]$ with $|m|>0$, $1\le r_1<\cdots<r_n$ (if $r_\alpha\ge l+2$, then interpret 2^{l+2-r_α} as 1), if $\Delta(m)$ has a term of the form $\eta\otimes\zeta_1^{j2^{l+1}}$ with $|\eta|>0$, then η projects to zero under $\bar{A}^*\to\bar{A}^*_{l-1}$. If $r_\alpha\le l+1$, then

$$\Delta(\zeta_{r_\alpha}^{k_\alpha2^{l+2-r_\alpha}})=\begin{cases}(\zeta_1^{2^{l+1}}\otimes1+1\otimes\zeta_1^{2^{l+1}})^{k_\alpha} & (r_\alpha=1),\\ (\zeta_2^{2^l}\otimes1+1\otimes\zeta_2^{2}+\zeta_1^{2^l}\otimes\zeta_1^{2^{l+1}})^{k_\alpha} & (r_\alpha=2),\\ \Big(\zeta_{r_\alpha}^{2^{l+2-r_\alpha}}\otimes1+1\otimes\zeta_{r_\alpha}^{2^{l+2-r_\alpha}} & \\ \quad+\sum_{p=2}^{r_\alpha-1}\zeta_{r_\alpha-p}^{2^{l+2-r_\alpha}}\otimes\zeta_p^{2^{l+2-p}}+\zeta_{r_\alpha-1}^{2^{l+2-r_\alpha}}\otimes\zeta_1^{2^{l+1}}\Big)^{k_\alpha} & (r_\alpha>2).\end{cases}$$

if $r_\alpha\ge l+2$, then

$$\Delta(\zeta_{r_\alpha}^{k_\alpha})=\Big(\zeta_{r_\alpha}\otimes1+1\otimes\zeta_{r_\alpha}+\sum_{p=2}^{r_\alpha-1}\zeta_{r_\alpha-p}\otimes\zeta_p^{2^{r_\alpha-p}}+\zeta_{r_\alpha-1}\otimes\zeta_1^{2^{r_\alpha-1}}\Big)^{k_\alpha}.$$

It follows that if $x\otimes\zeta_1^q\in\Delta(\zeta_{r_\alpha}^{k_\alpha2^{l+2-r_\alpha}})$, then q is a multiple of 2^{l+1} (we allow $q=0$) and x is of the form

$$\zeta_1^{\lambda_12^l}\zeta_{t_2}^{\lambda_22^{l+1-t_2}}\cdots\zeta_{t_p}^{\lambda_p2^{l+1-t_p}}$$

with $2\le t_2<\cdots<2_p$ (if $t_\alpha\ge l+1$, then $2^{l+1-t_\alpha}\equiv1$). This implies

$$\eta=\zeta_1^{u_12^l}\zeta_{s_2}^{u_22^{l+1-s_2}}\cdots\zeta_{s_q}^{u_q2^{l+1-s_q}}$$

with $2\le s_2<\cdots<s_q$ (if $s_\alpha\ge l+1$, then $2^{l+1-s_\alpha}\equiv1$). Since $|\eta|>0$, $u_\alpha>0$ for some α. From (6) (with l replaced by $l-1$) we see η projects to zero in $\bar{A}^*_{l-1}$. This proves $(*)$.

Similarly,

$$a\overline{\chi(i\,2^{l+1})}=a\overline{\chi(2i\,2^l)}=0 \quad \text{in } \bar{\Omega}'=\overline{A/A\cdot\bar{A}_{l-1}} \quad \text{for } a\in\bar{A}_{l-2}. \tag{$**$}$$

Let $\{\gamma_\alpha\}_{\alpha\ge0}$ ($\{\gamma'_\beta\}_{\beta\ge0}$) be a right A_l-base (A_{l-1}-base) for A such that $\bar{B}=\{\bar{\gamma}_\alpha\}_{\alpha\ge0}$ is the $\mathbb{Z}_2$-base for Ω in (7). Given $a\in\bar{A}_{l-2}$. Let $a\chi(i\,2^{l+1})=\sum_\lambda\gamma'_{j(\lambda)}a_{j(\lambda)}$ with

$\gamma'_{j(\lambda)} \in \{\gamma'_\beta\}_{\beta\ge 0}$ and $a_{j(\lambda)} \in A_{l-1}$. The result (**) implies $a_{j(\lambda)} \in \bar{A}_{l-1}$ for each λ (since $i>0$). Let

$$\gamma'_{j(\lambda)} = \sum_{\nu} \gamma_{j(\lambda,\nu)} b_{j(\lambda,\nu)}$$

with $\gamma_{j(\lambda,\nu)} \in \{\gamma_\alpha\}_{\alpha\ge 0}$ and $b_{j(\lambda,\nu)} \in A_l$. Then

$$a\chi(i\,2^{l+1}) = \sum_{\lambda,\nu} \gamma_{j(\lambda,\nu)} b_{j(\lambda,\nu)} a_{j(\lambda)}.$$

Since $a_{j(\lambda)} \in \bar{A}_{l-1}$, from formula (1) in Section 2 and the result (*) above we see

$$\overline{a(\chi(i\,2^{l+1})} \otimes \overline{\chi(j\,2^{l+1}))} = \sum_{\lambda,\nu} \bar{\gamma}_{j(\lambda,\nu)} \otimes b_{j(\lambda,\nu)} a_{j(\lambda)} \overline{\chi(j\,2^{l+1})} = 0.$$

This completes the proof of Proposition 3.4.

We conclude this section with the following corollary to Proposition 3.4 which is rather clear. In stating the corollary we note that if $a\in A_l$ with $|a|\le 2^{l-1}-1$, then $a\in A_{l-2}$, and so $\mathrm{Ext}^{s,t}_{A_{l-2}}(\mathbb{Z}_2,\mathbb{Z}_2)\cong \mathrm{Ext}^{s,t}_{A_l}(\mathbb{Z}_2,\mathbb{Z}_2)$ for $t-s\le 2^{l-1}-2$.

Corollary 3.5. (i) *If* $\sum_\nu [\alpha_{\nu_1}|\cdots|\alpha_{\nu_s}]$ *is a cocycle in* $F(A_l^*)$ *representing a non-zero class in* $\mathrm{Ext}^{s,*}_{A_l}(\mathbb{Z}_2,\mathbb{Z}_2)$ *and if* $|\alpha_{\nu_k}|\le 2^{l-1}-1$ *for all* ν_k, *then* $\sum_\nu [\alpha_{\nu_1}|\cdots|\alpha_{\nu_s}]\otimes(\zeta_1^{i2^{l+1}}\otimes\zeta_1^{j2^{l+1}})$ *is a cocycle in* $F(A_l^*)\otimes(\bar{\Omega}^*)^2$ *representing a non-zero class in* $\mathrm{Ext}^{s+2,*}_{A_l}((\bar{\Omega})^2,\mathbb{Z}_2)$ *where* $i>0$, $j>0$.

(ii) *Let* $R=\sum_\nu [\alpha_1|\cdots|\alpha_{\nu_s}]\otimes(\zeta_1^{i2^{l+1}}\otimes\zeta_1^{j2^{l+1}})$ *be as in* (i) *and let* $R_1=\sum_\mu [\beta_{\mu_1}|\cdots|\beta_{\mu_s}]\otimes(\eta_{u_{s+1}}\otimes\eta_{u_{s+2}})$ *be another cocycle in* $F(A_l^*)\otimes(\bar{\Omega}^*)^2$ *such that either* $\eta_{\mu_{s+1}}\otimes\eta_{\mu_{s+2}}\ne\zeta_1^{i2^{l+1}}\otimes\zeta_1^{j2^{l+1}}$ *for any* μ *or* R_1 *has a subsum of the form* $(\sum_\lambda [\beta_{\lambda_1}|\beta_{\lambda_2}|\cdots|\beta_{\lambda_s}])\otimes(\zeta_1^{p2^{l+1}}\otimes\zeta_1^{q2^{l+1}})$ *such that* $|\beta_{\lambda_k}|\le 2^{l-1}-1$ *for all* λ_k, $(p,q)\ne(i,j)$ *and* $\sum_\lambda [\beta_{\lambda_1}|\beta_{\lambda_2}\cdots|\beta_{\lambda_s}]$ *is a non-boundary cocycle. Then* $\{R\}\ne\{R_1\}$ *in* $\mathrm{Ext}^{s+2,*}_{A_l}((\bar{\Omega})^2,\mathbb{Z}_2)$.

4. Proof of Theorem 1.1

Let α, s, t, i, d_{i+1} and m be as in Theorem 1.1. These notations will be fixed throughout this section. We recall that d_{i+1} is the largest integer for which there are non-zero elements $a\in A_{i+1}$ such that $|a|=d_{i+1}$. From (6) of Section 3 we see

$$d_{i+1} = |\chi(2^{i+2}-1, 2^{i+1}-1,\ldots,3,1)| = i\,2^{i+3}+i+6.$$

By assumption $2^{m-1}>sd_{i+1}-t$ and $t-s\le 2^i-2$. Since $t-s>0$ and $\alpha\ne 0$, Adams vanishing theorem on $\mathrm{Ext}^{*,*}_A(\mathbb{Z}_2,\mathbb{Z}_2)$ [3] implies $t+3\ge 3s$. From these one easily verifies that

(a) 2^{m+1} is a positive multiple of 2^{i+2},

(b) $t+2^{m+1}>(2^{i+2}-1)(s+1)$, and

(c) $d_{i+1}>\frac{3}{2}(2^i-2+1)\ge t$.

To prove Theorem 1.1 we apply Theorem 2.1 by taking $\Gamma = A$ and $\Lambda = A_{i+1}$. By Proposition 3.1

$$\Omega^* = (A/A \cdot \bar{A}_{i+1})^* = \mathbb{Z}_2[\zeta_1^{2^{i+2}}, \zeta_2^{2^{i+1}}, \dots, \zeta_{i+2}^2, \zeta_{i+3}, \dots].$$

So $B = \{\chi(r_1, r_2, \dots) \mid r_j = k_j 2^{i+3-j}$ for $j \le i+2$ and $r_{i+k} \ge 0$ for $k \ge 3\}$ is a right A_{i+1}-base for A. Let $i^*: A^* \to A_{i+1}^*$ be dual to the inclusion $i: A_{i+1} \to A$ and let $\sigma: A_{i+1}^* \to A^*$ be defined by $\sigma(\zeta_1^{r_1}\zeta_2^{r_2}\cdots) = \zeta_1^{r_1}\zeta_2^{r_2}\cdots$. Then $i^*\sigma = 1_{A_{i+1}^*}$. By Proposition 2.4 there is a chain equivalence

$$f_1: F(A_{i+1}^*) \otimes \bar{\Omega}^* \to F^{(1)}/F^{(2)}$$

which is given by

$$f_1([\alpha_1 | \cdots | \alpha_s] \otimes \alpha_{s+1}) = [\sigma(\alpha_1) | \cdots | \sigma(\alpha_s) | \alpha_{s+1}]. \tag{8}$$

This formula is obtained by dualizing (2) in Section 2. With respect to the A_{i+1}-base B for A above we define

$$\bar{g}_2: \bar{B}(A_{i+1}) \otimes (\bar{\Omega})^2 \to \bar{B}^{(2)}/\bar{B}^{(1)}$$

by formula (3) in Section 2 and then take its $\mathbb{Z}_2$-dual

$$g_2: F^{(2)}/F^{(3)} \to F(A_{i+1}^*) \otimes (\bar{\Omega}^*)^2.$$

By Proposition 2.4, g_2 is a chain equivalence. It is not easy to write a formula for g_2. All we need about g_2 for what follows is Lemma 2.5, and we recall that there is a convention in the lemma which for the present case is the following. When we consider a cochain $[\alpha_1 | \cdots | \alpha_s]$ in $F(A^*)$ (or $F(A_{i+1}^*)$) or in $F^{(p)}/F^{(p+1)}$ the elements α_j will be monomials in the variables ζ_k.

We proceed to prove Theorem 1.1. If $s = 1$, then $\alpha = h_k$ for some k. Adams [2] has shown $h_k h_j^2 \ne 0$ if $j > k+2$. Since

$$2^{m-1} > sd_{i+1} - t = d_{i+1} - t \ge d_{i+1} - \tfrac{1}{2}(2^i - 2 + 1)$$

(by (c)) it follows that $m > k+2$. So $h_k h_m^2 \ne 0$. We may thus assume $s \ge 2$.

Let $\sum_\lambda [\alpha_{\lambda_1} | \cdots | \alpha_{\lambda_s}] \in F(A^*)^{s,t}$ be a cocycle representing the class α. Then

$$R = \sum_\lambda [\alpha_{\lambda_1} | \cdots | \alpha_{\lambda_s} | \zeta_1^{2^m} | \zeta_1^{2^m}] \in F(A^*)^{s+2, t+2^{m+1}}$$

is a cocycle representing αh_m^2. By (a), $\zeta_1^{2^m} \in \bar{\Omega}^*$. Since $2^i - 2 \ge t - s$ and $|\alpha_{\lambda_j}| \ge 1$, it follows that $2^i - 1 \ge |\alpha_{\lambda_j}|$ for all λ_j; so $\alpha_{\lambda_j} \in \sigma(\bar{A}_{i+1}^*)$. Therefore R lies in $F^{(2)}$ and its image $\bar{R}$ in $F^{(2)}/F^{(3)}$ is non-zero. By Lemma 2.5(i) the cocycle $g_2(\bar{R}) \in F(A_{i+1}^*) \otimes (\bar{\Omega}^*)^2$ is given by

$$g_2(\bar{R}) = \sum_\lambda [i^*(\alpha_{\lambda_1}) | \cdots | i^*(\alpha_{\lambda_s})] \otimes (\zeta_1^{2^m} \otimes \zeta_1^{2^m}) \tag{9}$$

and since $|\alpha_{\lambda_j}| \le 2^i - 1$, by Corollary 3.5(ii), it represents a non-zero class in

$$E_1^{2,s,t+2^{m+1}} = \mathrm{Ext}_{A_{i+1}}^{s+2,t+2^{m+1}}((\bar{\Omega})^2, \mathbb{Z}_2);$$

we denote this class by $\overline{\alpha h_m^2}$. To complete the proof of Theorem 1.1 it suffices to show that

(d) $d_1(x) \neq \overline{\alpha h_m^2}$ for any x in $E_1^{1,s,t+2^{m+1}} = \mathrm{Ext}_{A_{i+1}}^{s+1,t+2^{m+1}}(\bar{\Omega}, \mathbb{Z}_2)$, and

(e) $d_2(y) \neq \overline{\alpha h_m^2}$ for any y in $E_2^{0,s+1,t+2^{m+1}} \subset \mathrm{Ext}_{A_{i+1}}^{s+1,t+2^{m+1}}(\mathbb{Z}_2, \mathbb{Z}_2)$.

It is well known, by the May spectral sequence [10], that

$$\mathrm{Ext}_{A_{i+1}}^{\bar{s},\bar{t}}(\mathbb{Z}_2, \mathbb{Z}_2) = 0 \quad \text{for } \bar{t} > (2^{i+2}-1)\bar{s}.$$

From (b) we see $\mathrm{Ext}_{A_{i+1}}^{s+1,t+2^{m+1}}(\mathbb{Z}_2, \mathbb{Z}_2) = 0$. This proves (e).

It takes more work to prove (d). We need two lemmas. Let $Z = [\sigma(\theta_1) | \cdots | \sigma(\theta_s) | z]$ be a cochain in $F(A^*)^{s+1,t+2^{m+1}}$ with $\theta_j \in \bar{A}_{i+1}^*$, $z \in \bar{\Omega}^*$. Suppose $\delta(Z) = \sum_\lambda [\tau_{\lambda_1} | \cdots | \tau_{\lambda_{s+1}} | \tau_{\lambda_{s+2}}] \neq 0$ where δ is the coboundary homomorphism of $F(A^*)$. Since the coproduct $\Delta: A^* \to A^* \otimes A^*$ maps Ω^* to $A^* \otimes \Omega^*$ it follows that $\tau_{\lambda_{s+2}} \in \bar{\Omega}^*$ for all λ; so $\delta(Z) \in F^{(1)}$. Let ${}_2Z = \sum_\mu [\tau_{\mu_1} | \cdots | \tau_{\mu_{s+1}} | \tau_{\mu_{s+2}}]$ be the subsum of all $[\tau_{\mu_1} | \cdots | \tau_{\mu_{s+1}} | \tau_{\mu_{s+2}}]$ such that τ_{μ_j} annihilates $\bar{A}_{i+1}$ for only one $\mu_j \in \{\mu_1, \ldots, \mu_{s+1}\}$. Then ${}_2Z \in F^{(2)}$. Let ${}_2\bar{Z}$ be its image in $F^{(2)}/F^{(3)}$ and consider $g_2({}_2\bar{Z})$.

Lemma 4.1. (i) *If the sum $g_2({}_2\bar{Z})$ is non-zero and has a term of the form $[\eta_1 | \cdots | \eta_s] \otimes (\zeta_1^{p2^{i+2}} \otimes \zeta_1^{q2^{i+2}})$ with $(p+q)2^{i+2} = 2^{m+1}$ and $p2^{i+2} > sd_{i+1} - t$, then $\eta_j = \sigma(\theta_j)$ for all j, $z = \zeta_1^{k2^{i+2}} \zeta_2^{l2^{i+2}}$ for some k and $l > 0$ with $|z| = 2^{m+1}$ and $\zeta_1^{p2^{i+2}} \otimes \zeta_1^{q2^{i+2}} \in \Delta(\zeta_1^{k2^{i+2}} \zeta_2^{l2^{i+2}})$.*

(ii) *Conversely suppose $z = \zeta_1^{k2^{i+2}} \zeta_2^{l2^{i+2}}$ with $l > 0$ and $|z| = 2^{m+1}$. Then*

$$[\sigma(\theta_1) | \cdots | \sigma(\theta_s)] \otimes (\zeta_1^{(k+l)2^{i+3}} \otimes \zeta_1^{l2^{i+3}}) \in g_2({}_2\bar{Z})$$

and, if $k = l$,

$$[\sigma(\theta_1) | \cdots | \sigma(\theta_s)] \otimes (\zeta_1^{2^{m-1}} \otimes \zeta_1^{2^m + 2^{m-1}}) \in g_2({}_2\bar{Z}).$$

Lemma 4.2. (i) *Suppose $\zeta_1^{p2^{i+2}} \otimes \zeta_1^{q2^{i+2}} \in \Delta(\zeta_1^{k2^{i+2}} \zeta_2^{l2^{i+2}})$ and $(k+3l)2^{i+2} = 2^{m+1}$. If $p \geq q$ or if $p2^{i+2} = 2^{m-1}$ and $q2^{i+2} = 3 \cdot 2^{m-1}$, then $k \geq l$.*

(ii) *If $k - l > k' - l'$, then*

$$\zeta_1^{(k+l)2^{i+2}} \otimes \zeta_1^{l2^{i+3}} \notin \Delta(\zeta_1^{k'2^{i+2}} \zeta_2^{l'2^{i+2}}).$$

Proof of Lemma 4.1. Let $T(\mu) = [\tau_{\mu_1} | \cdots | \tau_{\mu_{s+1}} | \tau_{\mu_{s+2}}]$. Then $g_2({}_2\bar{Z}) = \sum_\mu g_2(\overline{T(\mu)})$. So

$$[\eta_1 | \cdots | \eta_s] \otimes (\zeta_1^{p2^{i+2}} \otimes \zeta_1^{q2^{i+2}}) \in g_2(\overline{T(\nu)}) \quad \text{for some } \nu.$$

Let ν_j be the only element in $\{\nu_1, \ldots, \nu_{s+1}\}$ such that τ_{ν_j} annihilates $\bar{A}_{i+1}$.

If $j \leq s$, then either

$$\tau_{\nu_{j-1}} \otimes \tau_{\nu_j} \in \Delta(\sigma(\theta_{j-1})), \qquad \tau_{\nu_k} = \sigma(\theta_k) \quad \text{for } k \leq j-2,$$

$$\tau_{\nu_k} = \sigma(\theta_{k-1}) \quad \text{for } j+1 \leq k \leq s+1 \qquad \text{and} \qquad \tau_{\nu_{s+2}} = z,$$

or

$$\tau_{v_j} \otimes \tau_{v_{j+1}} \in \Delta(\sigma(\theta_j)), \qquad \tau_{v_k} = \sigma(\theta_k) \quad \text{for } k \le j-1,$$

$$\tau_{v_k} = \sigma(\theta_{k-1}) \quad \text{for } j+2 \le k \le s+1 \text{ (if } j<s) \qquad \text{and} \qquad \tau_{v_{s+2}} = z.$$

Since $|\sigma(\theta_k)| \le d_{i+1}$ for all k it follows that both cases imply $\sum_{k=1}^{s+1} |\tau_{v_k}| \le sd_{i+1}$. By Lemma 2.5(ii) this would imply

$$[\eta_1 | \cdots | \eta_s] \otimes (\zeta_1^{p2^{i+2}} \otimes \zeta_1^{q2^{i+2}}) \notin g_2(\overline{T(v)}).$$

Therefore $j = s+1$. Then either

$$\tau_{v_s} \otimes \tau_{v_{s+1}} \in \Delta(\sigma(\theta_s)),$$

$$\tau_{v_k} = \sigma(\theta_k) \quad \text{for } k \le s-1 \qquad \text{and} \qquad \tau_{v_{s+1}} = z,$$

or

$$\tau_{v_{s+1}} \otimes \tau_{v_{s+2}} \in \Delta(z) \qquad \text{and} \qquad \tau_{v_j} = \sigma(\theta_j) \quad \text{for } 1 \le j \le s.$$

Since $g(\overline{T(v)}) \neq 0$, by Lemma 2.5(i), $\tau_{v_{s+1}} \in \bar{\Omega}^*$ and

$$g(\overline{T(v)}) = [\tau_{v_1} | \cdots | \tau_{v_s}] \otimes (\tau_{v_{s+1}} \otimes \tau_{v_{s+2}})$$
$$= [\eta_1 | \cdots | \eta_s] \otimes (\zeta_1^{p2^{i+2}} \otimes \zeta_1^{q2^{i+2}}).$$

By assumption, $p2^{i+2} > sd_{i+1} - t$ and $s \ge 2$. By (c), $d_{i+1} > t$. So $p2^{i+2} > d_{i+1}$ which implies $\tau_{v_s} \otimes \tau_{v_{s+1}} = \eta_s \otimes \zeta_1^{p2^{i+2}} \notin \Delta(\sigma(\theta_s))$. Hence

$$\tau_{v_{s+1}} \otimes \tau_{v_{s+2}} = \zeta_1^{p2^{i+2}} \otimes \zeta_1^{q2^{i+2}} \in \Delta(z) \tag{*}$$

with $|z| = (p+q)2^{i+2} = 2^{m+1}$ and $\eta_j = \tau_{v_j} = \sigma(\theta_j)$ for $1 \le j \le s$.

Since z is a monomial in $\Omega^* = \mathbb{Z}_2[\zeta_1^{2^{i+2}}, \zeta_2^{2^{i+1}}, \ldots, \zeta_{i+2}^2, \zeta_{i+3}, \ldots]$ it follows from formula (5) in Section 3 that z has to be of the form $\zeta_1^{k2^{i+2}} \zeta_2^{l'2^{i+1}}$ in order to have (*). We have $|z| = k2^{i+2} + 3l'2^{i+1} = 2^{m+1}$. Since $2^{i+2} \mid 2^{m+1}$, l' is even, say $l' = 2l$. So

$$z = \zeta_1^{k2^{i+2}} \zeta_2^{l2^{i+2}}.$$

l is positive because $p2^{i+2} > 0$, $q2^{i+2} > 0$ and

$$\Delta(\zeta_1^{2^{m+1}}) = 1 \otimes \zeta_1^{2^{m+1}} + \zeta_1^{2^{m+1}} \otimes 1.$$

This proves part (i).

To prove (ii) let $\delta([\sigma(\theta_1) | \cdots | \sigma(\theta_s)] = \sum_\lambda [\psi_{\lambda_1} | \cdots | \psi_{\lambda_{s+1}}]$ and let $\sum_v [\psi_{v_1} | \cdots | \psi_{v_{s+1}}]$ be the subsum of all $[\psi_{v_1} | \ldots | \psi_{v_{s+1}}]$ such that ψ_{v_j} annihilates $\bar{A}_{i+1}$ for exactly one $v_j \in \{v_1, \ldots, v_{s+1}\}$. Let

$$\Delta(z) = \Delta(\zeta_1^{k2^{i+2}} \zeta_2^{l2^{i+2}}) = \sum_{j=1}^{n} y_j' \otimes y_j'' + 1 \otimes z + z \otimes 1$$

with $y_j', y_j'' \in \bar{A}^*$. It is easy to see that $y_j', y_j'' \in \bar{\Omega}^*$ for all j. Then

$${}_2\bar{Z} = \sum_v [\psi_{v_1} | \cdots | \psi_{v_{s+1}} | z] + \sum_{j=1}^{n} [\sigma(\theta_1) | \cdots | \sigma(\theta_s) | y_j' | y_j''].$$

By Lemma 2.5(i),

$$g_2({}_2\bar{Z}) = \sum_{\nu} g_2([\psi_{\nu_1}|\cdots|\psi_{\nu_{s+1}}|z]) + \sum_{j=1}^{n} [\sigma(\theta_1)|\cdots|\sigma(\theta_s)]\otimes(y_j'\otimes y_j'').$$

It is easy to see that

$$\zeta_1^{(k+l)2^{i+2}}\otimes\zeta_1^{l2^{i+3}} = y_a'\otimes y_a'' \quad \text{for some } a$$

and that, if $k=l$,

$$\zeta_1^{2^{m-1}}\otimes\zeta_1^{3\cdot 2^{m-1}} = y_b'\otimes y_b'' \quad \text{for some } b.$$

Since $z\neq\zeta_1^{l2^{i+3}}$ and $z\neq\zeta_1^{3\cdot 2^{m-1}}$, it follows from Lemma 2.5(ii) that

$$[\sigma(\theta_1)|\cdots|\sigma(\theta_s)]\otimes(\zeta_1^{(k+l)2^{i+2}}\otimes\zeta_1^{l2^{i+3}})\notin g_2([\psi_{\nu_1}|\cdots|\psi_{\nu_{s+1}}|z])$$

and that, if $k=l$,

$$[\sigma(\theta_1)|\cdots|\sigma(\theta_s)]\otimes(\zeta_1^{2^{m-1}}\otimes\zeta_1^{3\cdot 2^{m-1}})\notin \sum_{\nu} g_2([\psi_{\nu_1}|\cdots|\psi_{\nu_{s+1}}|z]).$$

This implies the conclusion of part (ii). This completes the proof of Lemma 4.1. □

Proof of Lemma 4.2. We have

$$\Delta(\zeta_1^{k2^{i+2}}) = \sum_{\nu=0}^{k}\binom{k}{\nu}\zeta_1^{\nu 2^{i+2}}\otimes\zeta_1^{(k-\nu)2^{i+2}}$$

and

$$\Delta(\zeta_2^{l2^{i+2}}) = (\zeta_2^{2^{i+2}}\otimes 1 + 1\otimes\zeta_2^{2^{i+2}} + \zeta_1^{2^{i+2}}\otimes\zeta_1^{2^{i+3}})^l.$$

So $\zeta_1^{p2^{i+2}}\otimes\zeta_1^{q2^{i+2}} = \zeta_1^{(\nu+l)2^{i+2}}\otimes\zeta_1^{(k-\nu+2l)2^{i+2}}$ for some ν with $0\le\nu\le k$. Thus $p=\nu+l$ and $q=k-\nu+2$. If $\nu+l=p\ge q=k-\nu+2l$, then $0=k-2\nu+l\ge k-2k+l=l-k$, i.e., $k\ge l$. If $p2^{i+2}=2^{m-1}$ and $q2^{i+2}=3\cdot 2^{m-1}$, then $k-\nu+2l=3(\nu+l)$ which implies $k-l=4\nu\ge 0$, i.e., $k\ge l$. This proves (i).

To prove (ii) it suffices to show that if

$$\zeta_1^{\lambda 2^{i+2}}\otimes\zeta_1^{\mu 2^{i+3}}\in\Delta(\zeta_1^{k'2^{i+2}}\zeta_2^{l'2^{i+2}}),$$

then $\lambda-2\mu<k-l=(k+l)-2l$. As shown above we have $\lambda=\nu+l'$ and $2\mu=k'-\nu+2l'$ for some $\nu\le k'$. Then

$$\begin{aligned}\lambda-2\mu &= (\nu+l')-(k'-\nu+2l') = 2\nu-k'-l'\\ &\le 2k'-k'-l' = k'-l' < k-l.\end{aligned}$$

This proves (ii). □

Now we prove (d). Given any non-zero class x in

$$E_1^{1,s,t+2^{m+1}} = H^{s+1,t+2^{m+1}}(F^{(1)}/F^{(2)}) \cong \mathrm{Ext}_{A_{i+1}}^{s+1,t+2^{m+1}}(\bar{\Omega},\mathbb{Z}_2).$$

We have to show $d_1(x) \neq \overline{\alpha h_m^2}$.

By Proposition 2.4 and formula (8) x can be represented by a cocycle of the form

$$X = \sum_{\lambda} [\sigma(\theta_{\lambda_1}) | \cdots | \sigma(\theta_{\lambda_s}) | z_{\lambda_{s+1}}]$$

in $F^{(1)}/F^{(2)}$ where $\theta_{\lambda_j} \in \bar{A}_{i+1}^*$, $z_{\lambda_{s+1}} \in \bar{\Omega}^*$. We may assume

$$\text{Each subsum of } X \text{ which is a cocycle is not a boundary.} \tag{10}$$

We may consider X as in $F^{(1)} \subset F(A^*)$. Since X is a cocycle in $F^{(1)}/F^{(2)}$, $\delta(X) \in F^{(2)}$. Let $\overline{\delta(X)}$ be its image in $F^{(2)}/F^{(3)}$ and let

$$g_2(\overline{\delta(X)}) = \sum_{\nu} [\eta_{\nu_1} | \cdots | \eta_{\nu_s}] \otimes (w_{\nu_{s+1}} \otimes w_{\nu_{s+2}}).$$

$g_2(\overline{\delta(X)})$ is a cocycle in $F(A_{i+1}^*) \otimes (\bar{\Omega}^*)^2$ and, by definition, represents $d_1(x) \in \mathrm{Ext}_{A_{i+1}}^{s+2, t+2^{m+1}}((\bar{\Omega})^2, \mathbb{Z}_2)$.

If $w_{\nu_{s+1}} \otimes w_{\nu_{s+2}} \neq \zeta_1^{2^m} \otimes \zeta_1^{2^m}$ for any ν then, by Corollary 3.5(ii), $d_1(x) = \{g_2(\overline{\delta(X)})\} \neq \{g_2(\bar{R})\} = \overline{\alpha h_m^2}$ (we recall that $\overline{\alpha h_m^2}$ is represented by the cocycle $g_2(\bar{R})$ in (9)).

We may thus assume

(f) $\qquad w_{\nu_{s+1}} \otimes w_{\nu_{s+2}} = \zeta_1^{2^m} \otimes \zeta_1^{2^m}$ for some ν.

In this case we shall prove that the sum $g_2(\overline{\delta(X)})$ has a subsum of the form $(\sum_u \eta'_{u_1} | \cdots | \eta'_{u_s}) \otimes (\zeta_1^{p2^{i+2}} \otimes \zeta_1^{q2^{i+2}})$ such that $|\eta'_{u_j}| \leq 2^i - 1$ and $(p2^{i+2}, q2^{i+2}) \neq (2^m, 2^m)$ and $\sum_u [\eta'_{u_1} | \cdots | \eta'_{u_s}]$ is a non-boundary cocycle. This will imply $d_1(x) \neq \overline{\alpha h_m^2}$ again by Corollary 3.5(ii).

For each λ let $Z_\lambda = [\sigma(\theta_{\lambda_1}) | \cdots | \sigma(\theta_{\lambda_s}) | z_{\lambda_{s+1}}]$ and let ${}_2\bar{Z}_\lambda$ be as defined in (4.1). Then $X = \sum_\lambda Z_\lambda$ and

$$g_2(\overline{\delta(X)}) = \sum_{\lambda} g_2({}_2\bar{Z}_\lambda).$$

The assumption (f) means

$$[\eta_{\nu_1} | \cdots | \eta_{\nu_s}] \otimes (\zeta_1^{2^m} \otimes \zeta_1^{2^m}) \in g_2(\overline{\delta(X)})$$

which implies

$$[\eta_{\nu_1} | \cdots | \eta_{\nu_s}] \otimes (\zeta_1^{2^m} \otimes \zeta_1^{2^m}) \in g_2({}_2\bar{Z}_\lambda)$$

for some λ. Since $2^m > 2^{m-1} > sd_{i+1} - t$, by Lemma 4.1(i),

$$z_{\lambda_{s+1}} = \zeta_1^{k2^{i+2}} \zeta_2^{l2^{i+2}} \quad \text{with } l > 0,\ |z| = 2^{m+1}$$

and

$$\zeta_1^{2^m} \otimes \zeta_1^{2^m} \in \Delta(\zeta_1^{k2^{i+2}} \zeta_2^{l2^{i+2}}).$$

By Lemma 4.2(i) the latter implies $k \geq l$.

We now rewrite the cocycle X as

$$X=\sum_{\mu}[\sigma(\theta_{\mu_1})|\cdots|\sigma(\theta_{\mu_s})|z_{\mu_{s+1}}]+\sum_{a=1}^{n}[\sigma(\theta_1^{(a)})|\cdots|\sigma(\theta_s^{(a)})|\zeta_1^{k_a2^{i+2}}\zeta_2^{l_a2^{i+2}}]$$

where $k_a\geq l_a>0$, $(k_a+3l_a)2^{i+2}=2^{m+1}$ for all a, and for any μ the monomial $z_{\mu_{s+1}}$ is not of the form $\zeta_1^{k'2^{i+2}}\zeta_2^{l'2^{i+2}}$ with $k'\geq l'>0$ and $(k'+3l')2^{i+2}=2^{m+1}$. The result proved in the preceding paragraph shows that the second sum is not zero. Note that $\sum_{j=1}^{s}|\sigma(\theta_j^{(a)})|=t$. Since $|\sigma(\theta_j^{(a)})|\geq 1$ and $t-s\leq 2^i-2$ it follows that $|\sigma(\theta_j^{a})|\leq 2^i-1$ for all a and j. Let

$$Z_\mu=[\sigma(\theta_{\mu_1})|\cdots|\sigma(\theta_{\mu_s})|z_{\mu_{s+1}}]$$

and

$$Z_a=[\sigma(\theta_1^{(a)})|\cdots|\sigma(\theta_s^{(a)})|\zeta_1^{k_a2^{i+2}}\zeta_2^{l_a2^{i+2}}].$$

Then

$$g_2(\delta(X))=\sum_{\mu}g_2({}_2\bar{Z}_\mu)+\sum_{a=1}^{n}g_2({}_2\bar{Z}_a).$$

Let $D=\max_{1\leq a\leq n}\{k_a-l_a\}$. Then $D\geq 0$. We discuss in two cases: (i) $D=0$ and (ii) $D>0$.

Suppose $D=0$. Then $k_a2^{i+2}=l_a2^{i+2}=2^{m-1}$; so

$$Z_a=[\sigma(\theta_1^{(a)})|\cdots|\sigma(\theta_s^{(a)})|\zeta_1^{2^{m-1}}\zeta_2^{2^{m-1}}]$$

for all a. We may assume $[\sigma(\theta_1^{(a)}|\cdots|\sigma(\theta_s^{(a)})]\neq[\sigma(\theta_1^{(b)})|\cdots|\sigma(\theta_s^{(b)})]$ if $a\neq b$. By Lemma 4.1(ii), for each a,

$$[\sigma(\theta_1^{(a)})|\cdots|\sigma(\theta_s^{(a)})]\otimes(\zeta_1^{2^{m-1}}\otimes\zeta_1^{3\cdot 2^{m-1}})\in g_2({}_2\bar{Z}_a).$$

By assumption, $2^{m-1}>sd_{i+1}-t$. So, by Lemma 4.1(i) and Lemma 4.2(i),

$$[\sigma(\theta_1^{(a)})|\cdots|\sigma(\theta_s^{(a)})]\otimes(\zeta_1^{2^{m-1}}\otimes\zeta_1^{3\cdot 2^{m-1}})\begin{cases}\notin g_2({}_2\bar{Z}_b) & (b\neq a),\\ \notin g_2({}_2\bar{Z}_\mu) & (\text{all }\mu).\end{cases}$$

Thus $\sum_{a=1}^{n}[\sigma(\theta_1^{(a)})|\cdots|\sigma(\theta_s^{(a)})]\otimes(\zeta_1^{2^{m-1}}\otimes\zeta_1^{3\cdot 2^{m-1}})$ is a subsum of $g_2(\overline{\delta(X)})$. Note that $(2^{m-1},3\cdot 2^{m-1})\neq(2^m,2^m)$ and $|\sigma(\theta_j^{(1)})|\leq 2^i-1$ for all j, 2^{m-1} and $3\cdot 2^{m-1}$ are multiples of 2^{i+2} and, by (10), $\sum_{a=1}^{n}[\sigma(\theta_1^{(a)}|\cdots|\sigma(\theta_s^{(a)})]$ is a non-boundary cocycle.

Suppose $D>0$. We may assume that for some $n'\leq n$, $D=k_a-l_a$ for $1\leq a\leq n'$. It is easy to see that k_a and l_a are constants for $1\leq a\leq n'$. Let k and l be these two constants; so $D=k-l>0$. Then

$$Z_a=[\sigma(\theta_1^{(a)})|\cdots|\sigma(\theta_s^{(a)})|\zeta_1^{k2^{i+2}}\zeta_2^{l2^{i+2}}]$$

for $1\leq a\leq n'$. We may assume $[\sigma(\theta_1^{(a)})|\cdots|\sigma(\theta_s^{(a)})]\neq[\sigma(\theta_1^{(b)})|\cdots|\sigma(\theta_s^{(b)})]$ for $a\neq b$ $(1\leq a\leq n', 1\leq b\leq n')$. By Lemma 4.1(ii)

$$[\sigma(\theta_1^{(a)})|\cdots|\sigma(\theta_s^{(a)})]\otimes(\zeta_1^{(k+l)2^{i+2}}\otimes\zeta_1^{l2^{i+3}})\in g_2({}_2\bar{Z}_a).$$

$(k+l)^{i+2} - l2^{i+3} = (k-l)2^{i+2} > 0$ and $(k+l)2^{i+2} + l2^{i+3} = 2^{m+1}$ imply

$$(k+l)2^{i+2} > 2^m > 2^{m-1} > sd_{i+1} - t.$$

So, by Lemma 4.1(i) and Lemma 4.2(i), for $1 \le a \le n'$,

$$[\sigma(\theta_1^{(a)}) | \cdots | \sigma(\theta_s^{(a)})] \otimes (\zeta_1^{(k+l)2^{i+2}} \otimes \zeta_1^{l2^{i+3}}) \begin{cases} \notin g_2({}_2\bar{Z}_b) & (b \ne a, 1 \le b \le n'), \\ \notin g_2({}_2\bar{Z}_\mu) & (\text{all } \mu). \end{cases}$$

Since $k - l = D > k_a - l_a$ for $n'+1 \le a \le n$, it follows from Lemma 4.1(i) and Lemma 4.2(ii) that for each a with $1 \le a \le n'$

$$[\sigma(\theta_1^{(a)}) | \cdots | \sigma(\theta_s^{(a)})] \otimes (\zeta_1^{(k+l)2^{i+2}} \otimes \zeta_1^{l2^{i+3}}) \notin g_2({}_2\bar{Z}_b)$$

if $n'+1 \le b \le n$. Thus

$$\sum_{a=1}^{n'} [\sigma(\theta_1^{(a)}) | \cdots | \sigma(\theta_s^{(a)})] \otimes (\zeta_1^{(k+l)2^{i+2}} \otimes \zeta_1^{l2^{i+3}})$$

is a subsum of $g_2(\overline{\delta(X)})$. Note that $((k+l)2^{i+2}, l2^{i+3}) \ne (2^m, 2^m)$, $|\sigma(\theta_j^{(1)})| \le 2^i - 1$ for all j and, by (10), $\sum_{a=1}^{n'} [\sigma(\theta_1^{(a)}) | \cdots | \sigma(\theta_s^{(a)})]$ is a non-boundary cocycle.

This completes the proof of (d) and therefore Theorem 1.1.

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